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The use of Lie transformation groups in the solution of the coupled diffusion equation

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Abstract. The solutions of coupled, linear and nonlinear, diffusion equations in a semi-infinite medium, are derived using the method of continuous one-parameter point symmetry group invariants. Initially, conditions for both classical and non-classical Lie group invariants are developed and solutions are derived corresponding to linear coupled diffusion. The non-classical invariants are shown to be expressible in terms of a linear parabolic operator. In addition, solutions corresponding to constant, impulse and sinusoidally time varying condition are derived from the equations determining classical group invariance. Nonlinear cases are also considered. The classical symmetry groups for any form of diffusion matrix are presented. In addition, the point source solution for a power-law form governing the diffusion matrix is derived. It is a natural extension of the scalar analogue.

1. Introduction

The mathematical analysis presented in this paper is motivated by the problem of the coupled transport of heat, moisture and solute in a vertical column of unsaturated soil and the extent to which one-parameter Lie point symmetries may be used in the solution of the resulting system of one-dimensional partial differential equations. The need to study the symmetries of similar systems involving chemical reaction equations, even in linear form, has been expressed by Hill (1992). It is clear, that in such cases both the classical, non-classical and hidden symmetries (Guo and Abraham-Shrauner 1993) are not well understood.

The coupled diffusion discussed here occur in theories presented by Philip and De Vries (1957), De Vries (1958) and further discussed by many authors, for example, Jury, Letey and Stolzy (1987), where it is known that heat, moisture and solute will be transported under gradients of soil temperature, volumetric moisture content and solute concentration. Such effects are particularly noticeable under semi-arid or desert conditions where most moisture movement takes place in the vapour phase. The resulting simultaneous diffusion equations for a semi-infinite medium, describing the vertical transfer of both heat and moisture, in the absence of solutes, may be written in the simplified form of Jury, Letey and Stolzy (1987):

$$Z(x,t,Y,\dot{Y},Y',Y'') = \frac{\partial Y}{\partial t} - \frac{\partial}{\partial x}\left\{\Lambda(Y)\frac{\partial Y}{\partial x}\right\} = 0 \qquad Y = \sum_i e_i y_i \tag{1}$$

where $Y(x,t)$ is a vector $\{y_i(x,t)\}$ of soil temperature, moisture content and solute concentration values as a function of soil depth x and time t. Also, $\{e_i\}$ is a set of orthonormal vectors and $\Lambda(Y)$ is a non-singular, often diagonally dominant matrix, defining the diffusive properties of the soil.

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Equations in the general form of (1) have been studied by many authors, for example, Dayan and Gluekler (1982), Glazunov (1983) and Ghali (1986). However, the resulting analyses are usually complex, detailed and wholly numerical and attempts to present analytic properties of (1), even in the simplest of terms, are scarce (Baca *et al* 1978, Wiltshire 1992, 1993). In particular, the linearized form of (1)

$$Z(x,t,Y,\dot{Y},Y',Y'') = \frac{\partial Y}{\partial t} - \Lambda\frac{\partial^2 Y}{\partial x^2} = 0 \tag{2}$$

where Λ is constant has not been solved under a wide range of soil surface boundary conditions.

It is our aim here to obtain solutions for these equations, generated by determining the one-parameter transformation groups which leave (1) and (2) invariant. In recent years there has been much renewed interest (Chester 1977, Olver 1986, Stephani 1989, Hill 1992) in the method, initially developed by Lie, of continuous transformation groups as applied to the solution of differential equations. The main reasons for this has been the realization that the methodology, originally developed for ordinary differential equations, may be extended to the determination of analytic solutions to both linear and nonlinear partial differential equations. Clearly, this is an important development especially in the absence of general theories on nonlinear systems.

2. One-parameter Lie point transformations

Consider the one-parameter Lie point group of transformations defined by

$$x_1 = f(x,t,Y,\epsilon) \qquad t_1 = g(x,t,Y\epsilon) \qquad Y_1 = H(x,t,Y\epsilon) \tag{3}$$

where $H(x,t,Y,\epsilon)$ is a vector and where the infinitesimal form of (3) is:

$$\begin{aligned} x_1 &= x + \epsilon\xi(x,t,Y) + \mathrm{O}(\epsilon^2) \equiv x + \epsilon\mathcal{L}x + \mathrm{O}(\epsilon^2) \\ t_1 &= t + \epsilon\eta(x,t,Y) + \mathrm{O}(\epsilon^2) \equiv t + \epsilon\mathcal{L}t + \mathrm{O}(\epsilon^2) \\ Y_1 &= Y + \epsilon\pi(x,t,Y) + \mathrm{O}(\epsilon^2) \equiv Y + \epsilon\mathcal{L}Y + \mathrm{O}(\epsilon^2). \end{aligned} \tag{4}$$

The infinitesimal generator, $\mathcal{L}$ is defined by

$$\mathcal{L} = \xi(x,t,Y)\frac{\partial}{\partial x} + \eta(x,t,Y)\frac{\partial}{\partial t} + \pi(x,t,Y)\cdot\nabla \qquad \nabla = \sum_i e_i\frac{\partial}{\partial y_i} \tag{5}$$

and $\{e_i\}$ are orthogonal unit vectors. The link between the global and infinitesimal forms is defined by

$$\frac{\mathrm{d}x_1}{\mathrm{d}\epsilon} = \xi(x_1,t_1,Y_1) \qquad \frac{\mathrm{d}t_1}{\mathrm{d}\epsilon} = \eta(x_1,t_1,Y_1) \qquad \frac{\mathrm{d}Y_1}{\mathrm{d}\epsilon} = \pi(x_1,t_1,Y_1) \tag{6}$$

with, $x_1 = x$, $t_1 = t$ and $Y_1 = Y$ when $\epsilon = 0$. In addition, the condition that the solution $Y - \phi(x,t) = 0$ remains invariant under the transformations (4) is given by

$$\pi(x,t,Y) = \xi(x,t,Y)Y' + \eta(x,t,Y)\dot{Y}. \tag{7}$$

Moreover, the prolongation operator may be written as:

$$\mathcal{L}_E = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial t} + \pi \cdot \nabla + \pi_x \cdot \nabla_{Y'} + \pi_t \cdot \nabla_Y + \pi_{xx} \cdot \nabla_{Y''} \tag{8}$$

where

$$\pi_x = \pi' - \dot{Y}\eta' + Y' \cdot (\nabla\pi - \xi') - Y'(Y' \cdot \nabla)\xi - \dot{Y}(Y' \cdot \nabla)\eta \tag{9}$$

$$\pi_t = \dot{\pi} - Y'\dot{\xi} + \dot{Y} \cdot (\nabla\pi - \dot{\eta}) - Y'(\dot{Y} \cdot \nabla)\xi - \dot{Y}(\dot{Y} \cdot \nabla)\eta \tag{10}$$

$$\begin{aligned} \pi_{xx} = {} & \pi'' + Y' \cdot (2\nabla\pi' - \xi'') - \dot{Y}\eta'' + (Y' \cdot \nabla)^2\pi - 2Y'(Y' \cdot \nabla\xi') - 2\dot{Y}(Y' \cdot \nabla\eta') \\ & - Y'(Y' \cdot \nabla)^2\xi - \dot{Y}(Y' \cdot \nabla)^2\eta + Y'' \cdot [\nabla\pi - \nabla\xi Y' - \nabla\eta\dot{Y} - 2\xi' - 2Y'\nabla\xi] \\ & - 2\dot{Y}'(\eta' + Y' \cdot \nabla\eta) \end{aligned} \tag{11}$$

$$\nabla_{Y'} = \sum_i e_i \frac{\partial}{\partial y_i'} \qquad \nabla_{Y''} = \sum_i e_i \frac{\partial}{\partial y_i''}. \tag{12}$$

In cases where the partial differential equations are linear then it follows without loss in generality that $\xi = \xi(x, t)$, $\eta = \eta(x, t)$ and $\pi = \Gamma(x, t)Y$.

3. Classical and non-classical symmetry (the linear case)

The condition for invariance of equation (2) may be found by setting $\mathcal{L}_E Z = 0$ with the result that

$$\begin{aligned} \frac{\partial Y_1}{\partial t_1} - \Lambda \frac{\partial^2 Y_1}{\partial x_1^2} = {} & (\dot{\Gamma} - \Lambda\Gamma'')Y + (-\dot{\xi} - 2\Lambda\Gamma' + \Lambda\xi'')Y' + 2\Lambda\eta'\dot{Y}' \\ & + (2\Lambda\xi' + [\Gamma\Lambda] - \dot{\eta}\Lambda + \Lambda^2\eta'')Y'' = 0 \end{aligned} \tag{13}$$

where $[AB]$ is the commutator defined by:

$$[AB] = AB - BA. \tag{14}$$

The classical symmetry groups may be found by setting the coefficients of Y and partial derivatives equal to zero. Thus, it follows from the coefficient of $\partial^2 Y/\partial t \partial x$ that $\eta = \eta(t)$ and from the remaining coefficients of Y and partial derivatives that

$$\dot{\Gamma} - \Lambda\Gamma'' = 0 \tag{15}$$

$$-\dot{\xi} - 2\Lambda\Gamma' + \Lambda\xi'' = 0 \tag{16}$$

and

$$2\Lambda\xi' + [\Gamma\Lambda] - \dot{\eta}\Lambda = 0. \tag{17}$$

To find the non-classical symmetry groups it is first necessary to eliminate Y and $\partial Y/\partial t$ between equations (2) and (7) and then to use the result to eliminate Y and $\dot{Y}$ from (13). It follows that

$$(\dot{\Gamma} - \Lambda\Gamma'')\left(\frac{\Gamma}{\eta}\right)^{-1}\left\{\frac{\xi}{\eta}Y' + \Lambda Y''\right\} + (-\dot{\xi} - 2\Lambda\Gamma' + \Lambda\xi'')Y'$$
$$+(2\Lambda\xi' + [\Gamma\Lambda] - \dot{\eta}\Lambda + \Lambda^2\eta'')Y'' + 2\Lambda\eta'\left[\left\{\left(\frac{\Gamma}{\eta}\right)'\left(\frac{\Gamma}{\eta}\right)^{-1}\frac{\xi}{\eta} + \frac{\Gamma}{\eta} - \left(\frac{\xi}{\eta}\right)'\right\}Y'\right.$$
$$\left. + \left\{\left(\frac{\Gamma}{\eta}\right)'\left(\frac{\Gamma}{\eta}\right)^{-1}\Lambda - \frac{\xi}{\eta}\right\}Y''\right] = 0 \tag{18}$$

and so on equating coefficients of Y' and Y'' it is found that

$$\{\dot{Q} - \Lambda Q''\}q(Q)^{-1} - (\dot{q} - \Lambda q'') - 2\Lambda Q' = 0 \tag{19}$$

$$\{\dot{Q} - \Lambda Q''\}(Q)^{-1}\Lambda + [Q\Lambda] + 2\Lambda q' = 0 \tag{20}$$

where

$$Q = \frac{\Gamma}{\eta} \qquad q = \frac{\xi}{\eta}. \tag{21}$$

4. Classical symmetry groups for coupled linear diffusion

Consider now the solution of equations (15) to (17). Firstly, equation (17) may be differentiated respectively with respect to t and x to give

$$2\Lambda\dot{\xi}' = \Lambda\ddot{\eta} - [\dot{\Gamma}\Lambda] \qquad 2\Lambda\xi''' = -[\ddot{\Gamma}\Lambda] \tag{22}$$

and so on differentiating (16) with respect to x and substituting (22) it is found that:

$$\Lambda\Gamma(x,t) = \frac{I}{2}\left\{-\frac{1}{4}\frac{\mathrm{d}^2\eta}{\mathrm{d}t^2}x^2 + x\frac{\mathrm{d}\rho}{\mathrm{d}t}\right\} + I\sigma(t). \tag{23}$$

On substitution in (15) and using (16) it is clear that symmetry may be characterized by a five-parameter group, since

$$\eta(t) = \alpha + \mu t \tag{24}$$

$$\xi(x,t) = \lambda + \frac{\mu}{2}x - 2\beta t \tag{25}$$

$$\Lambda\Gamma(x,t) = (\kappa + \beta x)I. \tag{26}$$

Example (a). $\mu = 0$, $\kappa = 0$, $\beta = 0$.
In this case the group-invariant transformations are

$$x_1 = x + \epsilon\lambda \qquad t_1 = t + \epsilon\alpha \qquad Y_1 = Y \tag{27}$$

and the condition for invariance is:

$$\lambda\frac{\partial Y}{\partial x} + \alpha\frac{\partial Y}{\partial t} = 0 \Rightarrow Y = Y(\alpha x - \lambda t) \tag{28}$$

so that the solution of the coupled diffusion equation (2) may be written in the form:

$$Y(x,t) = \mathrm{e}^{(\alpha x - \lambda t)}Y_0 \tag{29}$$

where α and λ satisfy

$$(\Lambda\alpha^2 + \lambda)Y_0 = 0 \Rightarrow |\Lambda\alpha^2 + \lambda| = 0. \tag{30}$$

This corresponds to solutions presented by Wiltshire (1992) for the case when $\lambda = \mathrm{i}\omega$, corresponding to sinusoidal temperature and moisture variation at a soil surface $x = 0$.

Example (b). $\alpha = 0$, $\lambda = 0$, $\beta = 0$, $\kappa = 0$, $\mu = 1$.
This case of pure coupled diffusion also corresponds to a constant input at the boundary surface. This follows by applying the well known (Bluman and Kumei 1989) invariance condition that both the boundary condition and the set of points defining the boundary should remain invariant under the one-parameter group of transformations. Thus if $\boldsymbol{Y}(0,t) = \boldsymbol{a}$, $\boldsymbol{Y}(x,0) = \boldsymbol{b}$ are constant then $t = 0$ and $x = 0$ must be invariants of the symmetry group. Hence from (4) $\eta(0) = 0$ and $\xi(0,t) = 0$. Hence it follows from (24) and (25) that $\alpha = 0$, $\lambda = 0$ and $\beta = 0$. Furthermore, substitution of $\boldsymbol{Y}(0,t) = \boldsymbol{a}$, $\boldsymbol{Y}(x,0) = \boldsymbol{b}$ into (7) gives $\Gamma(0,t) = 0$ so that $\kappa = 0$.

Thus taking $\mu = 1$, and with the aid of (6), the transformation equations are

$$\frac{\mathrm{d}x_1}{\mathrm{d}\epsilon} = \xi(x_1, t_1) = \frac{x_1}{2} \qquad \frac{\mathrm{d}t_1}{\mathrm{d}\epsilon} = \eta(t_1) = t_1 \qquad \frac{\mathrm{d}\boldsymbol{Y}_1}{\mathrm{d}\epsilon} = 0 \tag{31}$$

and so, applying the initial conditions, $x_1 = x$; $t_1 = t$; $\boldsymbol{Y}_1 = \boldsymbol{Y}$ when $\epsilon = 0$, it may be shown that:

$$x_1 = x\mathrm{e}^{\epsilon/2} \qquad t_1 = t\mathrm{e}^{\epsilon} \qquad \boldsymbol{Y}_1 = \boldsymbol{Y}. \tag{32}$$

The corresponding differential equation (7) governing invariance becomes

$$0 = \frac{x}{2}\frac{\partial \boldsymbol{Y}}{\partial x} + t\frac{\partial \boldsymbol{Y}}{\partial t}. \tag{33}$$

This has the solution $\boldsymbol{Y} = \boldsymbol{Y}(s)$, where $s = xt^{-1/2}$, with the result that the coupled diffusion equation (2) becomes:

$$2\Lambda\frac{\mathrm{d}^2\boldsymbol{Y}}{\mathrm{d}s^2} + s\frac{\mathrm{d}\boldsymbol{Y}}{\mathrm{d}s} = 0. \tag{34}$$

This may be solved by setting:

$$\frac{\mathrm{d}\boldsymbol{Y}}{\mathrm{d}s} = \mathrm{e}^{-bs^2/4}\boldsymbol{Y}_0 \Rightarrow [\Lambda b - I]\boldsymbol{Y}_0 = 0 \tag{35}$$

so that $\boldsymbol{Y}_0$ is an eigenvector of Λ^{-1}, which means finally:

$$\boldsymbol{Y}(s) = \int_0^s \mathrm{e}^{-bs^2/4}\boldsymbol{Y}_0\,\mathrm{d}s + \text{constant}. \tag{36}$$

This solution is clearly identical in form to the familiar scalar solution.

Example (c). Impulsive input at the boundary surface.
In this case $\boldsymbol{Y}(x,0) = \delta(x)\boldsymbol{a}$, where $\boldsymbol{a}$ is a constant, so that (0,0) must be invariant under the coordinate transformation. Clearly from (4) and (24), when $t = 0$ then $\eta(0) = 0 \Rightarrow \alpha = 0$, thus $t_1 = t(1 + \epsilon\mu)$, so that $\mu = 0$. In addition since $x = 0$ and $t = 0$ simultaneously, equations (4) and (25) imply $\xi(0,0) = 0 \Rightarrow \lambda = 0$. Finally, on substitution of $\boldsymbol{Y}(x,0) = \delta(x)\boldsymbol{a}$ in (7) it follows that $\Gamma(x,0)\boldsymbol{Y}(x,0) = \xi(x,0)\partial\boldsymbol{Y}/\partial x \Rightarrow \kappa = 0$.

Hence with $\beta = 0$ this case becomes

$$\frac{\mathrm{d}x_1}{\mathrm{d}\epsilon} = \xi(x_1, t_1) = -2t_1 \qquad \frac{\mathrm{d}t_1}{\mathrm{d}\epsilon} = \eta(t_1) = 0 \qquad \frac{\mathrm{d}\boldsymbol{Y}_1}{\mathrm{d}\epsilon} = \Lambda^{-1}x_1\boldsymbol{Y}_1 \tag{37}$$

and, with the initial conditions $x_1 = x$; $t_1 = t$; $Y_1 = Y$, define the following transformations:

$$x_1 = x - 2\epsilon t \qquad t_1 = t \qquad Y_1 = \mathrm{e}^{\Lambda^{-1}(\epsilon x - \epsilon^2 t)} Y. \tag{38}$$

Furthermore, the condition for invariance is

$$\frac{\partial Y}{\partial x} = -\frac{x}{2t}\Lambda^{-1} Y \tag{39}$$

which may be solved with the substitution

$$Y(x,t) = \mathrm{e}^{-bx^2/4t} Y_0 g(t) \Rightarrow [bI - \Lambda^{-1}] Y_0 = 0 \tag{40}$$

so that Y_0 is an eigenvector of Λ^{-1}. Equation (40) also satisfies the coupled diffusion equation (2), provided that

$$Y(x,t) = t^{-1/2} \mathrm{e}^{-bx^2/4t} Y_0. \tag{41}$$

This is also identical to the familiar scalar solution when an impulse boundary condition is applied at $x = 0$.

5. Classical symmetry groups for coupled nonlinear diffusion

The condition for invariance of equation (1) may be found by setting $\mathcal{L}_E Z = 0$, which leads to the relationship

$$\pi_t = (\pi \cdot \nabla)\Lambda Y'' + Y'(\pi \cdot \nabla)\nabla\Lambda Y' + \Lambda \pi_{xx} + (\pi_x \cdot \nabla)\Lambda Y' + (Y' \cdot \nabla)\Lambda \pi_x. \tag{42}$$

Therefore, on expansion,

$$\begin{aligned}
\dot{\pi} - Y'\dot{\xi} + \dot{Y} \cdot (\nabla\pi - \dot{\eta}) - Y'(\dot{Y} \cdot \nabla)\xi - \dot{Y}(\dot{Y} \cdot \nabla)\eta \\
= (\pi \cdot \nabla)\Lambda\Lambda^{-1}[\dot{Y} - (Y' \cdot \nabla\Lambda)Y'] + Y'(\pi \cdot \nabla)\nabla\Lambda Y' \\
+ \Lambda \left\{\pi'' + Y' \cdot (2\nabla\pi' - \xi'') - \dot{Y}\eta'' + (Y' \cdot \nabla)^2\pi - 2Y'(Y' \cdot \nabla\xi')\right. \\
\left. - 2\dot{Y}(Y' \cdot \nabla\eta') - Y'(Y' \cdot \nabla)^2\xi - \dot{Y}(Y' \cdot \nabla)^2\eta - 2\dot{Y}'(\eta' + Y' \cdot \nabla\eta)\right\} \\
+ [\dot{Y} - (Y' \cdot \nabla\Lambda)Y'] \cdot [\nabla\pi - \nabla\xi Y' - \nabla\eta\dot{Y} - 2\xi' - 2Y'\nabla\xi] \\
+ \left\{(\pi' - \dot{Y}\eta' + Y' \cdot (\nabla\pi - \xi') - Y'(Y' \cdot \nabla)\xi - \dot{Y}(Y' \cdot \nabla)\eta) \cdot \nabla\right\}\Lambda Y' \\
+ (Y' \cdot \nabla)\Lambda\left\{\pi' - \dot{Y}\eta' + Y' \cdot (\nabla\pi - \xi') - Y'(Y' \cdot \nabla)\xi - \dot{Y}(Y' \cdot \nabla)\eta\right\}.
\end{aligned} \tag{43}$$

In order to find the classical symmetry groups, the coefficients must be equated to give the following conditions:

(1) constants: $\dot{\pi} = \Lambda \pi''$

(2) $\boldsymbol{Y}'$: $-\boldsymbol{Y}'\dot{\xi} = \Lambda \boldsymbol{Y}' \cdot [2\nabla \pi' - \xi''] + (\pi' \cdot \nabla)\Lambda \boldsymbol{Y}' + (\boldsymbol{Y}' \cdot \nabla)\Lambda \pi'$

(3) $\dot{\boldsymbol{Y}}$: $-\dot{\eta}\dot{\boldsymbol{Y}} = (\pi \cdot \nabla \Lambda)\Lambda^{-1}\dot{\boldsymbol{Y}} - \Lambda \dot{\boldsymbol{Y}}\eta'' - 2\xi'\dot{\boldsymbol{Y}}$

(4) $\dot{\boldsymbol{Y}}, \boldsymbol{Y}'$: $-\boldsymbol{Y}'(\dot{\boldsymbol{Y}} \cdot \nabla)\xi = -2\Lambda \dot{\boldsymbol{Y}}(\boldsymbol{Y}' \cdot \nabla \eta') - (\boldsymbol{Y} \cdot \nabla)\xi \boldsymbol{Y}' - 2\dot{\boldsymbol{Y}}(\boldsymbol{Y}' \cdot \nabla)\xi$
$\qquad -\dot{\boldsymbol{Y}}\eta' \cdot \nabla \Lambda \boldsymbol{Y}' - \boldsymbol{Y}'\nabla \Lambda \dot{\boldsymbol{Y}}\eta'$

(5) $\dot{\boldsymbol{Y}}, \dot{\boldsymbol{Y}}$: $-\dot{\boldsymbol{Y}}(\dot{\boldsymbol{Y}} \cdot \nabla)\eta = -\dot{\boldsymbol{Y}} \cdot \nabla \eta \dot{\boldsymbol{Y}}$

(6) $\boldsymbol{Y}', \boldsymbol{Y}'$: $0 = -(\pi \cdot \nabla \Lambda)\Lambda^{-1}(\boldsymbol{Y}' \cdot \nabla)\Lambda \boldsymbol{Y}' + \boldsymbol{Y}'(\pi \cdot \nabla)\nabla \Lambda \boldsymbol{Y}'$
$\qquad +\Lambda(\boldsymbol{Y}' \cdot \nabla)^2\pi - 2\Lambda \boldsymbol{Y}'(\boldsymbol{Y}' \cdot \nabla)\xi' + (((\boldsymbol{Y}' \cdot \nabla)\pi) \cdot \nabla)\Lambda \boldsymbol{Y}'$

Alternatively, this may be written as:

$0 = ((\boldsymbol{Y}' \cdot \nabla)[(\pi \cdot \nabla \Lambda)\Lambda^{-1}])\Lambda \boldsymbol{Y}' + \Lambda(\boldsymbol{Y}' \cdot \nabla)^2\pi - 2\Lambda \boldsymbol{Y}'(\boldsymbol{Y}' \cdot \nabla)\xi'$

(7) $\boldsymbol{Y}', \dot{\boldsymbol{Y}}, \dot{\boldsymbol{Y}}$: $0 = -\Lambda \dot{\boldsymbol{Y}}(\boldsymbol{Y}' \cdot \nabla)^2\eta + (\boldsymbol{Y}' \cdot \nabla)\Lambda \boldsymbol{Y}'\nabla \eta \dot{\boldsymbol{Y}}$
$\qquad -\dot{\boldsymbol{Y}}(\boldsymbol{Y}' \cdot \nabla)\eta \nabla \Lambda \boldsymbol{Y}' - \boldsymbol{Y}'\nabla \Lambda \dot{\boldsymbol{Y}}(\boldsymbol{Y}' \cdot \nabla)\eta$

(8) $\boldsymbol{Y}', \boldsymbol{Y}', \boldsymbol{Y}'$: $0 = -\Lambda \boldsymbol{Y}'(\boldsymbol{Y}' \cdot \nabla)^2\xi + [(\boldsymbol{Y}' \cdot \nabla)\Lambda \boldsymbol{Y}'] \cdot \{\nabla \xi \boldsymbol{Y}' + 2(\boldsymbol{Y}' \cdot \nabla)\xi\}$
$\qquad -\boldsymbol{Y}'(\boldsymbol{Y}' \cdot \nabla)\xi \nabla \Lambda \boldsymbol{Y}' - (\boldsymbol{Y}' \cdot \nabla)\Lambda \boldsymbol{Y}'(\boldsymbol{Y}' \cdot \nabla)\xi$

(9) $\dot{\boldsymbol{Y}}'$: $0 = -2\Lambda \dot{\boldsymbol{Y}}'\eta'$

(10) $\dot{\boldsymbol{Y}}', \boldsymbol{Y}'$: $0 = -2\Lambda \dot{\boldsymbol{Y}}'\boldsymbol{Y}'\nabla \eta.$

It is easy to show that conditions (4), (5), (7), (8), (9) and (10) are automatically satisfied whenever $\eta = \eta(t)$ and $\xi = \xi(x, t)$. In addition, condition (3) may be substituted into (6) with the result that

$$\Lambda(\boldsymbol{Y}' \cdot \nabla)^2\pi = 0 \tag{44}$$

and

$$\pi \cdot \nabla \Lambda = \Lambda(2\xi' - \dot{\eta}). \tag{45}$$

In cases when $(2\xi' - \dot{\eta}) = 0$, and for all $\Lambda(\boldsymbol{Y})$, condition (3) gives

$$\pi(x, t, \boldsymbol{Y}) = 0 \tag{46}$$

$$\xi = \xi(x, t, \boldsymbol{Y}) = \frac{\gamma}{2}x + \kappa \tag{47}$$

$$\eta = \eta(x, t, \boldsymbol{Y}) = \gamma t + \delta. \tag{48}$$

Alternatively, when $(2\xi' - \dot{\eta}) \neq 0$ then (44) is satisfied when

$$\pi = \frac{1}{m}(\boldsymbol{Y} + \beta) \tag{49}$$

where m and β are independent of y_i. Hence, using condition (3), it is found that Λ has the particular form:

$$\Lambda = A\prod_i (y_i + \beta_i)^{R_i} \tag{50}$$

where A is a constant matrix and

$$\sum_i R_i = m(2\xi' - \dot{\eta}). \tag{51}$$

In this case it may be shown conditions (1) and (2) will also be satisfied when

$$\xi = \xi(x, t, Y) = \lambda x + \kappa \tag{52}$$

$$\eta = \eta(x, t, Y) = \gamma t + \delta \tag{53}$$

$$\pi = (Y + \beta)\frac{(2\lambda - \gamma)}{\sum_i R_i}. \tag{54}$$

Consider, as an example, the source solution of

$$\dot{Y} = \frac{\partial}{\partial x}\left\{A \prod_i (y_i)^{R_i} \frac{\partial Y}{\partial x}\right\} \tag{55}$$

so that

$$Y(x, 0) = Y_0 \delta(x) \tag{56}$$

where the Dirac delta function satisfies

$$\delta(\lambda x) = \lambda^{-1}\delta(x). \tag{57}$$

Following the method of Hill (1992), we notice that equations (55) and (56) remain invariant under the transformations

$$x_1 = e^{\epsilon} x \qquad t_1 = e^{(\sum_i R_i + 2)\epsilon} t \qquad Y_1 = e^{-\epsilon} Y. \tag{58}$$

It follows that

$$\xi(x, t, Y) = x \qquad \eta(x, t, Y) = \left(\sum_i R_i + 2\right)t \qquad \pi = -Y \tag{59}$$

is a subset of (52) to (54). Using the condition for invariance (7) and (57), the functional form of $Y(x, t)$ with the associated similarity variable ω is given by

$$Y(x, t) = \frac{\phi(\omega)}{t^n} \qquad \omega = \frac{x}{t^n} \qquad n = \frac{1}{\left(2 + \sum_i R_i\right)}. \tag{60}$$

If we now substitute these into (55) it is found that

$$A \prod_i \phi_i^{R_i} \frac{d\phi}{d\omega} + n\omega\phi = a. \tag{61}$$

On setting $a = 0$ it may be shown that (66) has the solution

$$\phi(\omega) = \left[r + \frac{s\omega^2}{2}\right]^p b \tag{62}$$

where

$$p \sum_i R_i = 1. \tag{63}$$

Furthermore, A and b satisfy the eigenvalue relationship

$$A \prod_i b_i s b = -\left[\frac{\sum_i R_i}{2 + \sum_i R_i}\right] b. \tag{64}$$

This solution is similar in form to a well known scalar solution (Hill 1992).

6. Conclusions

In many previous studies of coupled diffusion, much of the analysis has been numerical (for example, Dayan and Gluekler 1982, Glazunov 1983 and Ghali 1986), with almost no analytical discussion in the literature. In addition, numerical investigation has not centred on the extent to which the system is linear or not, even though linear discussion can sometimes be adequate. It has been shown here, using the method of continuous, classical one-parameter point symmetry, how solutions for linear coupled diffusion with three separate boundary conditions are intimately related. Such solutions are shown to be natural extensions of the analogous linear scalar equations. It thus follows that an analytical sensitivity investigation of the relative importance of the elements of Λ is now possible for a wide range of boundary conditions, including the case of trickle irrigation (Ghali 1986). In cases where nonlinearity is important, solutions have been found whenever the coefficients of the Λ involve powers of temperature, moisture content and solute concentrations. This form for the Λ matrix is not unlike that encountered in reality. The solution presented as an example corresponds to an impulsive boundary condition at the soil surface and is an extension of a similar solution presented for the scalar nonlinear diffusion equation. Although the method of group invariance is powerful, its limitations are also transparent, especially when dealing with the classical cases of advection diffusion. In addition, it appears that there are no hidden symmetries. It may be concluded that a broader class of transformation (perhaps Backlund) needs to be considered in order to extract maximum analytical information from these equations.

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